1$^{\text{st}}$ GIMO 2021

Gaussian Curvature

Day I Problems

Problem 1. In an acute-angled triangle ABC with $AB \neq AC$, let P, Q be points on the segment BC, such that $CA = CP$ and $BA = BQ$. Let ω be the circumcircle of triangle ABC and AP, AQ intersect ω at points X and Y, other than A. The perpendiculars from X, Y onto line BC meet ω at X', Y', respectively.

How many pairs of points (M, N) (with M on AP and N on AQ) exist such that line MN is parallel to BC, and points M, N, X', Y' lie on the same circle?

Problem 2. A crazy biologist has $n \geq 3$ Petri dishes arranged in a line. Initially, some of them contain one bacterium each, while the rest are empty. The biologist then performs the following experiments:

> Firstly, he makes each bacterium divide into 2, and in each of the resulting pairs, one bacterium remains in its Petri dish and the other moves to the next one.

> Then, if some Petri dish contains at least 2 bacteria, the conditions there become unsuitable for living, and thus, one of them dies while one of them moves to the next Petri dish. In any case, a bacterium that leaves the last Petri dish dies.

> The experiment ends when the conditions in all Petri dishes are suitable for living.

Help the biologist find the least positive integer k (which may depend on n) such that after k experiments, the arrangement of bacteria in the Petri dishes is always the same as before experiments.

Problem 3. Determine all functions f mapping positive reals to positive reals such that

$$f(x)f(x + 2f(y)) = xf(x + y) + f(x)f(y),$$

for all positive reals x, y.
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